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LETTER TO THE EDITOR

Singularity structure, symmetries and integrability of generalized Fisher-type nonlinear diffusion equation

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Abstract

In this Letter, the integrability aspects of a generalized Fisher-type equation with modified diffusion in $(1+1)$ and $(2+1)$ dimensions are studied by carrying out a singularity structure and symmetry analysis. It is shown that the Painlevé property exists only for a special choice of the parameter ($m = 2$). A Bäcklund transformation is shown to give rise to the linearizing transformation to the linear heat equation for this case ($m = 2$). A Lie symmetry analysis also picks out the same case ($m = 2$) as the only system among this class having a nontrivial infinite-dimensional Lie algebra of symmetries and that the similarity variables and similarity reductions lead in a natural way to the linearizing transformation and physically important classes of solutions (including known ones in the literature), thereby giving a group theoretical understanding of the system. For nonintegrable cases in $(2 + 1)$ dimensions, associated Lie symmetries and similarity reductions are indicated.

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The Fisher-type reaction–diffusion equation with quadratic nonlinearity and modified diffusion of the form

$$u_t - \triangle u - \frac{m}{1-u}(\nabla u)^2 - u(1-u) = 0 \qquad (1)$$

where $u(t, x, y)$ is some kinetic variable, $\triangle$ and ∇ are Laplacian and gradient operators respectively and the subscript denotes partial differentiation with respect to time, is an important physical system appearing in many areas of physics and biology [1–4]. The case $m = 0$ is the classical Fisher equation and $m = 2$ corresponds to real systems such as bacterial colony growth [5]. It has been known for some time that the $m = 2$ case can be mapped onto the linear heat equation through a specific transformation [6]. Recently Brazhnik and Tyson [7] have discussed five interesting classes of travelling wave solutions and static structures in the two-dimensional version of the $m = 2$ case of equation (1). In this Letter, we wish to point out that a Painlevé singularity structure analysis [8] picks out the $m = 2$ case for both the $(1+1)$ and $(2+1)$ dimensions as the only system for which the partial differential equation (1) is free from movable critical singular manifolds satisfying the Painlevé (P-) property. More interestingly, we point out that the Bäcklund transformation deduced from the Laurent expansion gives rise to the linearizing transformation in a natural way. Similarly, a Lie symmetry analysis singles out the $m = 2$ case in equation (1) as the only system possessing a nontrivial infinite-dimensional Lie algebra of symmetries and that the similarity variables and similarity reductions give rise to the linearizing transformation and several physically interesting solutions, including the travelling wave solutions, static structures and so on known in the literature, in an automatic way. For the $m \neq 2$ cases, one obtains restricted classes of invariant solutions, including propagating pulses and fronts of special type.

To start with, we consider the $(1+1)$-dimensional case of equation (1). Locally expanding in the neighbourhood of the non-characteristic singular manifold $\phi(x, t) = 0$, $\phi_x, \phi_t \neq 0$ in the form of a Laurent series [8]

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+p} \tag{2}$$

one finds that the possible values of the power of the leading-order term are

(i) $p = -2$

(ii) $p = \dfrac{1}{1-m} \qquad m \neq 1$

(iii) $p = 0$.

One can easily check that for all these three leading-order cases, only for the value $m = 2$ is the solution free from movable critical singular manifolds. One finds that in the case $p = -1$, the leading-order coefficient u_0 is an arbitrary function in addition to the arbitrary singular manifold $\phi(x, t)$. In the case of the other two leading orders, for $m = 2$ one obtains only one arbitrary function without the introduction of movable singular manifolds and so they can be considered as corresponding to special solutions. Thus equation (1) in the one-dimensional case is found to satisfy the P-property for $m = 2$ and is expected to be integrable. For all other choices of m, except for certain special cases of the constrained singular manifold $\phi_t - \phi_{xx} = 0$, the solution exhibits the presence of a movable critical singular manifold, and so the system is of non-Painlevé type and so nonintegrable. Extending this analysis to the (2 + 1)-dimensional case also one obtains essentially the same conclusion (except for certain cases of the constrained manifold $\phi_t - \phi_{xx} - \phi_{yy} = 0$).

Now in the Laurent expansion (2) if we cut off the series at the 'constant' level term, that is $j = -p$ for the leading order $p = 1/(1-m) = -1$, $m = 2$, so that

$$u = \frac{u_0}{\phi} + u_1 \tag{3}$$

and demand that if u_1 is a solution of (1) for the case $m = 2$ then u is also a solution, one essentially has an auto-Bäcklund transformation. Here u_0 and ϕ satisfy a set of coupled partial differential equations (pdes) arising from equation (1) on using the transformation (3). Starting from the trivial solution, $u_1 = 0$ of equation (1), one can check that the equations for u_0 and ϕ in equation (3) are consistent for the choice $u_0 = \phi$, giving rise to the new solution $u = 1$, which is indeed an exact solution of (1). Now taking $u_1 = 1$ as the new seed solution, one can check from equations satisfied by u_0 and ϕ that

$$u_0 = -1 \qquad \phi_t - \phi_{xx} - \phi + 1 = 0. \tag{4}$$

Defining now $\phi = 1 + \chi$, one obtains the linear heat equation

$$\chi_t - \chi_{xx} - \chi = 0. \tag{5}$$

Thus the transformation

$$u = 1 - \frac{1}{1 + \chi} \tag{6}$$

where χ satisfies the linear heat equation (5), is the linearizing transformation for equation (1) in (1 + 1) dimensions for the choice $m = 2$, which is similar to that considered in [6]. Here we have given an interpretation for the transformation in terms of the Bäcklund transformation. Similar analysis holds good in the case of (2 + 1) dimensions of equation (1) also and the same transformation (6) linearizes equation (1) for $m = 2$ as well, where χ satisfies the two-dimensional linear heat equation $\chi_t - \chi_{xx} - \chi_{yy} - \chi = 0$.

Now let us consider the invariance of equation (1) under the one-parameter continuous Lie group of transformations. First we consider the (1 + 1)-dimensional case,

$$u_t - u_{xx} - \frac{m}{1 - u} u_x^2 - u + u^2 = 0. \tag{7}$$

Considering the infinitesimal transformation

$$\begin{aligned} &x \longrightarrow X = x + \varepsilon\xi(t, x, u) \qquad && t \longrightarrow T = t + \varepsilon\tau(t, x, u) \\ &u \longrightarrow U = u + \varepsilon\phi(t, x, u) \qquad && \varepsilon \ll 1 \end{aligned} \tag{8}$$

one can check that the invariance analysis of equation (7) singles out the special value $m = 2$ as the only choice having nontrivial Lie point symmetries of the form

$$\tau = a \qquad \xi = b \qquad \phi = c(t, x)(1 - u)^2 \tag{9}$$

where a, b are arbitrary constants and $c(t, x)$ is any solution of the linear heat equation

$$c_t - c_{xx} - c = 0. \tag{10}$$

The corresponding symmetry algebra is of the form

$$[X_1, X_2] = 0 \qquad [X_1, X_c] = X_{c_t} \qquad [X_2, X_c] = X_{c_x} \tag{11}$$

where

$$X_1 = \partial_t \qquad X_2 = \partial_x \qquad \text{and} \qquad X_c = c(t, x)(1 - u)^2\partial_u \tag{12}$$

and is infinite dimensional in nature. However, for all other values of m ($m \neq 2$) one obtains only the trivial translation symmetries

$$\tau = a \qquad \xi = b \qquad \phi = 0. \tag{13}$$

Solving now the characteristic equation associated with (9) for $m = 2$,

$$\frac{\mathrm{d}t}{a} = \frac{\mathrm{d}x}{b} = \frac{\mathrm{d}u}{c(t, x)(1 - u)^2} \tag{14}$$

one obtains the similarity variables

$$z = ax - bt \qquad u = 1 - \frac{a}{a + \bar{w}(z) + \int c(t, x)\,\mathrm{d}t} \tag{15}$$

where $\bar{w}$ satisfies the similarity-reduced ordinary differential equation (ode) of the form

$$a^2\bar{w}'' + b\bar{w}' + \bar{w} = 0. \tag{16}$$

Equation (16) is in fact the form of the linear heat equation in terms of the wave variable z. Further, one can consider the quantity

$$\frac{1}{a}\left[w(z) + \int c(t, x)\,\mathrm{d}t\right] = \chi \tag{17}$$

where χ satisfies equation (5), so that the similarity variable u in (15) is nothing but the linearizing transformation (6). It has now been given a group theoretical interpretation.

Further, since the general solution of (16) is

$$\bar{w} = I_1 \mathrm{e}^{m_1 z} + I_2 \mathrm{e}^{m_2 z} \qquad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a^2}}{2a^2} \tag{18}$$

where I_1 and I_2 are integration constants, we can write the solution to the original pde as

$$u = \begin{cases} 1 - \dfrac{a}{a + I_1 \mathrm{e}^{m_1(ax-bt)} + I_2 \mathrm{e}^{m_2(ax-bt)} + \int c(t,x)\,\mathrm{d}t} & b^2 - 4a^2 > 0 \\ 1 - \dfrac{a}{a + \mathrm{e}^{p(ax-bt)}\,(I_1 + I_2(ax-bt)) + \int c(t,x)\,\mathrm{d}t} & b^2 - 4a^2 = 0 \\ 1 - \dfrac{a}{a + \mathrm{e}^{p(ax-bt)}\,(I_1 \cos q(ax-bt) + I_2 \sin q(ax-bt)) + \int c(t,x)\,\mathrm{d}t} & \\ & b^2 - 4a^2 < 0 \end{cases} \tag{19}$$

with $p = -b/2a^2$, $q = \sqrt{4a^2 - b^2}/2a^2$. In the special case $c(t,x) = 0$ one obtains all the interesting travelling wave solutions and stationary structures discussed in [7].

For all other values of m ($\neq 2$), one obtains from the symmetries (13) the similarity variables $z = ax - bt$ and $u = w(z)$, leading to an ode of the form

$$a^2 \bar{w}\bar{w}'' - ma^2 \bar{w}'^2 + b\bar{w}\bar{w}' - (1-\bar{w})\bar{w}^2 = 0 \tag{20}$$

with $\bar{w} = 1 - w$. The above equation (20) is in general of non-Painlevé type when $a, b \neq 0$ except for $m = 0$ and $b = 5/\sqrt{6}$ under proper rescaling [9]. For the other choices one can solve for the case $b = 0$ in terms of elliptic function solutions including the limiting case of a solitary pulse, which are all of static form.

Now we extend our above analysis to the (2 + 1)-dimensional case,

$$u_t - u_{xx} - u_{yy} - \frac{m}{1-u}(u_x^2 + u_y^2) - u + u^2 = 0. \tag{21}$$

Under the infinitesimal transformation

$$\begin{aligned} &x \longrightarrow X = x + \varepsilon\xi(t,x,y,u) \qquad y \longrightarrow Y = y + \varepsilon\eta(t,x,y,u) \\ &t \longrightarrow T = t + \varepsilon\tau(t,x,y,u) \qquad u \longrightarrow U = u + \varepsilon\phi(t,x,y,u) \quad \varepsilon \ll 1 \end{aligned} \tag{22}$$

the invariance analysis separates out the special value of $m = 2$ for which equation (21) possesses the following Lie point symmetries:

$$\tau = a \qquad \xi = b_3 y + b_4 \qquad \eta = -b_3 x + d_4 \qquad \phi = c(t,x,y)(1-u)^2 \tag{23}$$

where b_3, b_4 and d_4 are arbitrary constants and $c(t,x,y)$ is the solution of the two-dimensional linear heat equation

$$c_t - c_{xx} - c_{yy} - c = 0. \tag{24}$$

However, for all other choices of m ($\neq 2$) we obtain

$$\tau = a \qquad \xi = b_3 y + b_4 \qquad \eta = -b_3 x + d_4 \qquad \phi = 0. \tag{25}$$

As earlier, for $m = 2$, the similarity variables are found by solving the characteristic equation associated with the symmetries (23). They are

$$\begin{aligned} &z_1 = \frac{b_3}{2}(x^2 + y^2) + b_4 y - d_4 x \qquad z_2 = -t - \frac{a}{b_3}\sin^{-1}\left(\frac{d_4 - b_3 x}{\sqrt{d_4^2 + 2b_3 z_1 + b_4^2}}\right) \\ &u = 1 - \frac{a}{w(z_1,z_2) + \int c(t,x,y)\,\mathrm{d}t} \end{aligned} \tag{26}$$

where w satisfies the similarity-reduced (1 + 1)-dimensional pde

$$w_{z_2} + 2b_3 w_{z_1} + (2b_3 z_1 + b_4^2 + d_4^2) w_{z_1 z_1} + \frac{a^2 w_{z_2 z_2}}{2b_3 z_1 + b_4^2 + d_4^2} + w - a = 0. \quad (27)$$

The above equation is nothing but the linear heat equation in terms of the variables z_1 and z_2. Here too as in equation (15) one can consider the similarity form (26) in two dimensions to obtain the linear heat equation

$$\chi_t - \chi_{xx} - \chi_{yy} - \chi = 0 \qquad \chi = \frac{1}{a}\left[w(z_1, z_2) + \int c(t, x, y)\,\mathrm{d}t\right] \quad (28)$$

so that the transformation for the variable u can again be interpreted as the linearizing transformation as in the one-dimensional case from a group theoretical point of view.

Carrying out again a Lie symmetry analysis on equation (27), one obtains the similarity variables

$$\begin{aligned} &\zeta = \bar{z}_1 \qquad w = a + \mathrm{e}^{(c_1\bar{z}_2/c_3)}\left[f(\zeta) + \frac{1}{c_3}\int \hat{c}_2(\bar{z}_1, \bar{z}_2)\mathrm{e}^{(-c_1\bar{z}_2/c_3)}\,\mathrm{d}\bar{z}_2\right] \\ &\bar{z}_1 = 2b_3 z_1 + b_4^2 + d_4^2 \qquad \bar{z}_2 = z_2 \qquad b_3, d_4 \neq 0 \end{aligned} \quad (29)$$

where f satisfies the linear second-order ode of the form

$$\zeta^2 f'' + \zeta f' + (A + B\zeta) f = 0 \qquad A = (ac_1/2b_3c_3)^2 \quad B = (1 + c_1/c_3)/4b_3^2 \quad (30)$$

with the prime denoting differentiation w.r.t. ζ and $\hat{c}_2(\bar{z}_1, \bar{z}_2)$ any solution of the transformed version of equation (27), $\hat{c}_{2\bar{z}_2} + 4b_3^2\hat{c}_{2\bar{z}_1} + 4b_3^2\bar{z}_1\hat{c}_{2\bar{z}_1\bar{z}_1} + \frac{a^2}{\bar{z}_1}\hat{c}_{2\bar{z}_2\bar{z}_2} + \hat{c}_2 = 0$. The function $\hat{c}_2(\bar{z}_1, \bar{z}_2)$ arises because of the linear nature of equation (27), and we can take $\hat{c}_2(\bar{z}_1, \bar{z}_2) = 0$ as well. However when $\hat{c}_2(\bar{z}_1, \bar{z}_2)$ is taken as non-zero, we wish to note that the solution to equation (21) as given below in equation (32) is richer in structure. We will therefore assume $\hat{c}_2(\bar{z}_1, \bar{z}_2)$ as nonzero.

As the general solution to equation (30) can be expressed in terms of cylindrical functions of the form

$$f = I_1 Z_1\big(2\sqrt{B\zeta}\big) + I_2 Z_2\big(2\sqrt{B\zeta}\big) \quad (31)$$

where the Z_i, $i = 1, 2$, are the two linearly independent cylindrical functions and I_1 and I_2 are arbitrary constants, the invariant solution to the (2 + 1)-dimensional pde (21) is written as

$$\begin{aligned} u = 1 - a\Bigg[a + \mathrm{e}^{(c_1\bar{z}_2/c_3)}\Bigg(& I_1 Z_1\big(2\sqrt{B\bar{z}_1}\big) + I_2 Z_2\big(2\sqrt{B\bar{z}_1}\big) \\ & - \frac{1}{c_3}\int \hat{c}_2\big(\bar{z}_1, \bar{z}_2\big)\mathrm{e}^{(c_1\bar{z}_2/c_3)}\,\mathrm{d}\bar{z}_2\Bigg) + \int c(t, x, y)\,\mathrm{d}t\Bigg]^{-1}. \end{aligned} \quad (32)$$

Next, proceeding to the special case, $b_3 = 0$ and $d_4 = 0$ in equation (26), and carrying out an analysis as above, we obtain the solution to the original pde (21) as

$$u = \begin{cases} 1 - a\Big\{a + \exp\Big[-k\Big(\dfrac{a}{b_4}x - t\Big)\Big]\Big[I_1 \cos\big(\sqrt{k_1}c_5 b_4 y\big) + I_2 \sin\big(\sqrt{k_1}c_5 b_4 y\big) \\ + \displaystyle\int \frac{\hat{c}_3(z_1, z_2)}{c_5}\mathrm{e}^{kz_2}\,\mathrm{d}z_2\Big] + \int c(t, x, y)\,\mathrm{d}t\Big\}^{-1} \qquad k_1 < 0 \\ 1 - a\Big\{a + \exp\Big[-k\Big(\dfrac{a}{b_4}x - t\Big)\Big]\Big[I_1 \mathrm{e}^{\sqrt{k_1}c_5 b_4 y} + I_2 \mathrm{e}^{-\sqrt{k_1}c_5 b_4 y} \\ + \displaystyle\int \frac{\hat{c}_3(z_1, z_2)}{c_5}\mathrm{e}^{kz_2}\,\mathrm{d}z_2\Big] + \int c(t, x, y)\,\mathrm{d}t\Big\}^{-1} \qquad k_1 > 0 \\ 1 - a\Big\{a + \exp\Big[-k\Big(\dfrac{a}{b_4}x - t\Big)\Big]\Big[I_1 c_5 b_4 y + I_2 \\ + \displaystyle\int \frac{\hat{c}_3(z_1, z_2)}{c_5}\mathrm{e}^{kz_2}\,\mathrm{d}z_2\Big] + \int c(t, x, y)\,\mathrm{d}t\Big\}^{-1} \qquad k_1 = 0 \end{cases} \tag{33}$$

where the parameter $k_1 = \frac{1}{b_4^2 c_5^2}\big[k - \big(\frac{ak}{b_4}\big)^2 - 1\big]$ with $k = -c_2/c_5$ and $z_1 = b_4 y,\ z_2 = \frac{a}{b_4}x - t$. Here $c_2,\ c_4,\ c_5$ are arbitrary constants of integration and $\hat{c}_3(z_1, z_2)$ is any solution of the equation $\hat{c}_{3z_2} + b_4^2 z_1 \hat{c}_{3z_1z_1} + \frac{a^2}{b_4^2}\hat{c}_{3z_2z_2} + \hat{c}_3 = 0$, which also arises from the symmetry of the linear equation (27) with $b_3 = d_4 = 0$. Again the presence of the additional function $\hat{c}_3(z_1, z_2)$ in the solution (33) leads to a more general form.

Equations (32) and (33) contain a large class of interesting solutions of various types including travelling wave solutions and static patterns. Particularly, we can easily check that all the five classes of travelling wave solutions discussed in [7] can be derived from equation (33) for particular choices of the constants involved along with the specific choice that the functions $\hat{c}_3(z_1, z_2) = 0$ and $c(t, x, y) = 0$. Specifically, the simplest *travelling wave solution*

$$u = 1 - \frac{1}{1 + A\exp\big[-k\big(\frac{a}{b_4}x - t\big) \pm \sqrt{k_1}c_5 b_4 y\big]} \qquad k_1 > 0 \tag{34}$$

can be constructed by assuming either $I_1 = 0$ or $I_2 = 0$. However, if we choose $I_1 = I_2\ (\neq 0)$ in equation (33), we obtain the *V-wave* solution

$$u = 1 - \frac{1}{1 + A\exp\big[-k\big(\frac{a}{b_4}x - t\big)\big]\cosh\big(\sqrt{k_1}c_5 b_4 y\big)} \qquad k_1 > 0. \tag{35}$$

On the other hand for $I_2 = 0$ and $k_1 < 0$ in equation (33), we obtain the *wave front oscillating* in space:

$$u = 1 - \frac{1}{1 + A\exp\big[-k\big(\frac{a}{b_4}x - t\big)\big]|\cos\big(\sqrt{k_1}c_5 b_4 y\big)|}. \tag{36}$$

The case $k_1 = 0$, $I_1 = 0$ describes a travelling plane wave, but when $I_1 \neq 0$ and $I_2 = 0$ we obtain an inhomogeneous solution

$$u = 1 - \frac{1}{1 + A|y|\exp\big[-k\big(\frac{a}{b_4}x - t\big)\big]} \tag{37}$$

which is nothing but the *separatrix* solution. Finally one more choice exists for positive k_1 and $I_1 = -I_2$:

$$u = 1 - \frac{1}{1 + A\exp\big[-k\big(\frac{a}{b_4}x - t\big)\big]|\sinh\big(\sqrt{k_1}c_5 b_4 y\big)|}. \tag{38}$$

This is the *Y-wave* solution. In each of the above solutions A is a positive constant. Several static structures can also be obtained as limiting cases of the above solutions (32) and (33).

Proceeding in a similar fashion for m other than 2 in equation (21), we obtain the similarity-reduced variables

$$z_1 = \frac{b_3}{2}(x^2+y^2) + b_4 y - d_4 x \qquad z_2 = -t - \frac{a}{b_3}\sin^{-1}\left(\frac{d_4 - b_3 x}{\sqrt{d_4^2 + 2b_3 z_1 + b_4^2}}\right) \tag{39}$$
$$u = w(z_1, z_2) \qquad (b_3, d_4 \neq 0)$$

along with the reduced pde of the form

$$w_{z_2} + 2b_3 w_{z_1} + (2b_3 z_1 + b_4^2 + d_4^2) w_{z_1 z_1} + \frac{a^2 w_{z_2 z_2}}{2b_3 z_1 + b_4^2 + d_4^2} + \frac{m}{1-w}$$
$$\times\left[(2b_3 z_1 + b_4^2 + d_4^2) w_{z_1}^2 + \frac{a^2 w_{z_2}^2}{2b_3 z_1 + b_4^2 + d_4^2}\right] + w - w^2 = 0. \tag{40}$$

On carrying out a similarity reduction, equation (40) reduces to an ode

$$4b_3^2\left[\zeta f'' + \left(f' + \frac{m\zeta}{1-f} f'^2\right)\right] + f - f^2 = 0 \qquad ' = \mathrm{d}/\mathrm{d}\zeta \tag{41}$$

where

$$\zeta = \bar{z}_1 \quad w = f(\zeta) \quad \bar{z}_1 = 2b_3 z_1 + b_4^2 + d_4^2 \tag{42}$$

giving rise to static structures in (x, y) variables. This equation is found to be nonintegrable, in general. However, if we consider for the special case $b_3 = 0,\ d_4 = 0$, the similarity variables become $z_1 = b_4 y,\ z_2 = \frac{a}{b_4}x - t,\ u = w(z_1, z_2)$; equation (40) on one more reduction gives the ode

$$\begin{aligned} &Df'' + \frac{Dm}{1-f} f'^2 - c_1 f' + f(1-f) = 0 \\ &D = \left(\frac{a^2}{b_4^2}{c_1}^2 + b_4^2 {c_2}^2\right) \qquad ' = \mathrm{d}/\mathrm{d}\zeta \end{aligned} \tag{43}$$

with $\zeta = -c_1(\frac{a}{b_4}x - t) + c_2 b_4 y$ and $w = f(\zeta)$, giving rise to plane wave solutions. This equation is of non-Painlevé type, except for the choice $m = 0$ in which case equation (43) reduces to the form (20) with $m = 0$.

On the other hand with the choice $b_3 = 0$ alone, the similarity variables become $z_1 = d_4 x - b_4 y,\ z_2 = ax - b_4 t$ and $u = w(z_1, z_2)$. Correspondingly equation (40) on a further reduction reduces to an ode

$$Af_1'' + Bf_1' - \frac{Am}{f_1} f_1'^2 - f_1 + f_1^2 = 0 \qquad ' = \mathrm{d}/\mathrm{d}\zeta \tag{44}$$

with $f_1 = 1 - f,\quad A = a^2(c_1^2 d_4^2 + c_2^2 b_4^2),\ B = -d_4 b_4(c_1 + c_2)$ and $\zeta = ac_2(d_4 x - b_4 y) - d_4(c_1 + c_2)(ax - b_4 t),\ w = f(\zeta)$. However, the system is found to possess elliptic function solutions including the limiting case of the solitary pulse for particular choices of the constants involved. More details of these results will be published elsewhere.

Throughout our above analysis we have made use of the computer program MUMATH [10] to determine the symmetries.

To conclude, the generalized Fisher-type equation with modified diffusion in (1 + 1) and (2 + 1) dimensions has been found to be integrable only for the special case $m = 2$ via both the singularity structure and symmetry analysis. Moreover we have also shown that a Bäcklund transformation gives rise to the linearizing transformation to the heat equation for the integrable case. Besides, a Lie symmetry analysis leads in a natural way to the linearizing transformation and physically important classes of solutions through similarity variables and similarity reductions, thereby giving a group theoretical understanding of the system.

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